# Definitions

## Orthogonality of state bases

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## Coordinate Representations

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## Projection Operators

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|  | 1. P |

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# Useful Identities

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## Properties of spherical harmonics:

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|  | 1. D |

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|  | 1. P |

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|  | 1. P |

## Properties of spherical Bessel functions:

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Note that the spherical Bessel functions are real-valued for real arguments.

## Properties of delta functions:

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Properties of plane waves

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|  | 1. D |

(Rayleigh expansion)

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|  | 1. ? |

(Unsöld’s theorem)

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# Proofs

## Projection of plane wave in spherical wave basis:

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Expanding the plane wave in spherical harmonics:

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## Projection of plane wave in offset spherical wave basis:

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Change variables:

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Expanding the plane wave in spherical harmonics:

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## Proof of Rayleigh Formula:

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We begin with the plane wave projected onto the spherical wave basis:

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We insert a complete set of states in the spherical wave basis:

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Since this is equal to

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## Completeness of spherical wave basis

Assert that

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Since we know

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Inserting the complete set:

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And the identity is proven.

## Unitariness of spherical wave projection operator:

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## Normalization of the plane waves

In order for the k-space basis vectors to be orthogonal:

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We insert the complete set:

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Applying the expansion of the plane waves in spherical waves:

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So the proper normalization of the plane waves in order to preserve this relation is:

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## Orthonormality of spherical harmonics

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Since the Kronecker delta function is defined as:

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And by the properties of the associated Legendre polynomials:

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Therefore

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And therefore the identity:

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Is proven.

## Unitariness of the plane wave basis:

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## Completeness of the k-space projection operator

Postulate:

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Since we know that

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Insert the postulated projection operator:

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Expanding the exponentials in spherical harmonics:

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And therefore the identity is proven.